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Two Real Critical Constraints for Real Parameter Margin Computation

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A new approach to computing real parameter margins of stabilized dynamical systems with multilinearly uncertain parameters is presented. A concept of two real critical constraints is introduced to solve the problem of determining the largest stable hypercube in parameter space that touches the stability boundary on one of its corners. The proposed approach is based on sufficient conditions for checking for critical instability only in the corner directions of the parameter space hypercube. Several examples are used to illustrate the proposed concept and approach.

I. Introduction

THIS paper is concerned with the problem of computing the structured singular values μ for uncertain dynamical systems.[1] In particular, the problem of computing ∞-norm real parameter margins (or real μ) is investigated, which is of much current research interest.[2-14] The real μ problem is essentially the same as the problem of determining the largest stable hypercube in the uncertain parameter space. We exploit the concept of separating the real and imaginary parts of a characteristic polynomial equation. The resulting two equations are referred to as the two real critical constraints, and the vertex (or corner) property of each constraint equation is utilized.

The paper is organized as follows: In Sec. II, we introduce the two real critical constraints, the two-constraint real μ, and the single-constraint real μ. In Sec. III, we show that the single-constraint real μ always attains its value at a corner of the parameter space hypercube of multilinearly uncertain systems at a given frequency. Section III contains the main result: a sufficient condition for the critical instability to occur at a corner of the parameter space hypercube of multilinearly uncertain systems. In Sec. IV, we apply the results of Secs. II and III to the problem of computing real parameter margins of several different types of characteristic polynomial. In Sec. V, several examples in the literature are used to illustrate the proposed concept and approach.

II. Two Real Critical Constraints

Consider a characteristic polynomial $\phi(s; p)$ with the real uncertain parameter vector

$$p = (p_1, p_2, \ldots, p_\ell)$$

where

$$\underline{p}_i \le p_i \le \bar{p}_i, \qquad i = 1, 2, \ldots, \ell$$

and $\underline{p}_i$ and $\bar{p}_i$ are, respectively, the prescribed lower and upper bounds of the ith element of the uncertain parameter vector p. The symbol $(p_1, p_2, \ldots, p_\ell)$ denotes a column vector in this paper; that is, $(p_1, p_2, \ldots, p_\ell) \equiv [p_1\ p_2 \ldots p_\ell]^T$.

The normalized uncertain parameter vector $\delta \in \mathfrak{D}$ is then defined as

$$\delta = (\delta_1, \delta_2, \ldots, \delta_\ell)$$

and the parameter space hypercube $\mathfrak{D}$ is defined as

$$\mathfrak{D} := \{\delta : -1 \le \delta_i \le 1, \qquad i = 1, \ldots, \ell\}$$

where

$$\delta_i = \frac{2(p_i - p_{0i})}{\bar{p}_i + \underline{p}_i} \tag{1}$$

and

$$p_{0i} = (\bar{p}_i + \underline{p}_i)/2, \qquad i = 1, \ldots, \ell$$

The nominal system of $\delta = 0$ is assumed to be asymptotically stable.

For a characteristic polynomial of the form $\phi(s; \delta)$, we have the following lemma.

Lemma 1: For any nth-order polynomial in s of the form $\phi(s; \delta)$ with the uncertain parameter vector $\delta \in \mathfrak{D}$, there exists an $n \times n$ rational matrix $M(s)$ and a diagonal matrix $\Delta \in X$ such that

$$\phi(s; \delta) = \phi(s; 0) \det[I + M(s)\Delta] \tag{2}$$

and

$$X := \{\Delta : \Delta = \operatorname{diag}(\delta_i I_i), \qquad i = 1, \ldots, \ell\} \tag{3}$$

where I_i denotes an $m_i \times m_i$ identity matrix and $\sum_{i=1}^{\ell} m_i = n$.

Proof: See Appendix.

According to Lemma 1, the critical stability constraint equation

$$\phi(j\omega; \delta) = 0 \tag{4}$$

simply becomes

$$\det[I + M(j\omega)\Delta] = 0 \tag{5}$$

since $\phi(j\omega; 0) \ne 0$ for all ω.

Definition 1: The real parameter robustness measure $\kappa(\omega)$ and the real structured singular value measure $\mu(\omega)$ associated with the critical constraint Eq. (5) are defined as

$$\kappa(\omega) \equiv 1/\mu(\omega)$$
$$:= \inf_{\Delta \in X} \{\kappa : \det[I + M(j\omega)\Delta] = 0,\ \bar{\sigma}(\Delta) \le \kappa\}$$
$$:= \sup_{\Delta \in X} \{\kappa : \det[I + M(j\omega)\Delta] \ne 0,\ \bar{\sigma}(\Delta) \le \kappa\}$$

where X is the set of all repeated blocks defined as Eq. (3) and $\bar{\sigma}(\Delta)$ denotes the largest singular value of Δ. The real parame-

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ter margin κ^* and the associated real structured singular value μ^* are then defined as

$$\kappa^* \equiv 1/\mu^* := \inf_{\omega} \kappa(\omega) \tag{6}$$

and the corresponding uncertain parameter vector is called the critical parameter vector and denoted by δ^*.

The critical stability constraint Eq. (4) or (5) is a complex constraint. We now exploit the idea of separating the real and imaginary parts of the constraint Eq. (4) as follows:

$$\mathrm{Re}[\phi(j\omega;\delta)] = f_1(\omega)\phi_1(\omega;\delta) = 0 \tag{7a}$$

$$\mathrm{Im}[\phi(j\omega;\delta)] = f_2(\omega)\phi_2(\omega;\delta) = 0 \tag{7b}$$

where $f_1(\omega)$ and $f_2(\omega)$ are polynomials that are independent of δ and $\phi_1(\omega;0) \neq 0$ and $\phi_2(\omega;0) \neq 0$ for all $\omega \geq 0$.

According to Lemma 1, there exist real rational matrices $M_1(\omega)$ and $M_2(\omega)$ such that

$$\phi_i(\omega;\delta) = \phi_i(\omega;0)\det[I + M_i(\omega)\Delta_i], \qquad i = 1, 2 \tag{8}$$

Consequently, we have the following two real critical constraints

$$f_1(\omega)\det[I + M_1(\omega)\Delta_1] = 0 \tag{9a}$$

$$f_2(\omega)\det[I + M_2(\omega)\Delta_2] = 0 \tag{9b}$$

with

$$X_i := \{\Delta_i : \Delta_i = \mathrm{diag}(\delta_{ij}I_{ij}), \quad \delta_{ij} \in \mathcal{R}, \quad j = 1, \ldots, \ell_i\} \tag{10}$$

and $\{\delta_{1j}, j = 1, \ldots, \ell_1\}$ and $\{\delta_{2j}, j = 1, \ldots, \ell_2\}$ are two subsets of $\{\delta_i, i = 1, \ldots, \ell\}$, and I_{ij} is an $m_{ij} \times m_{ij}$ identity matrix with $\sum_{j=1}^{\ell_i} m_{ij} = n_i$, $i = 1, 2$.

Polynomials with coefficients linearly dependent on uncertain parameters δ_i can be expressed in a form with rank-one matrices $M_1(\omega)$ and $M_2(\omega)$. This result is given as the following lemma.

Lemma 2: The critical stability constraint (4) of an interval polynomial or a polytopic polynomial can be expressed as two real critical constraints of the form (9) with rank-one matrices $M_1(\omega)$ and $M_2(\omega)$.

Proof: See Appendix.

Definition 2: A frequency at which the two real critical constraints (9) reduce to a single constraint is called the degenerate frequency.

Note that the real non-negative roots of the polynomials $f_1(\omega)$ and $f_2(\omega)$ of (9) are the degenerate frequencies, which cause isolated discontinuities in $\mu(\omega)$.

Definition 3: The two-constraint real μ measure, associated with the constraints (9), is defined as

$$1/\mu_{12}(\omega) := \inf_{\Delta \in X} \{\bar{\sigma}[\mathrm{diag}(\Delta_1,\Delta_2)] : \det[I + M_1\Delta_1] = 0 \quad \text{and} \quad \det[I + M_2\Delta_2] = 0\} \tag{11}$$

The single-constraint real μ measures $\mu_1(\omega)$ and $\mu_2(\omega)$, associated with each constraint in (9), are defined as

$$1/\mu_1(\omega) := \inf_{\Delta_1 \in X_1} \{\bar{\sigma}(\Delta_1) : \det[I + M_1\Delta_1] = 0\} \tag{12}$$

$$1/\mu_2(\omega) := \inf_{\Delta_2 \in X_2} \{\bar{\sigma}(\Delta_2) : \det[I + M_2\Delta_2] = 0\} \tag{13}$$

The real μ measure is related to the two-constraint real μ measure and the single-constraint real μ measures at each frequency ω, as follows:

$$\mu(\omega) = \begin{cases} \mu_{12}(\omega) & \text{if } f_1(\omega) \neq 0, f_2(\omega) \neq 0 \\ \mu_1(\omega) & \text{if } f_1(\omega) \neq 0, f_2(\omega) = 0 \\ \mu_2(\omega) & \text{if } f_1(\omega) = 0, f_2(\omega) \neq 0 \end{cases} \tag{14}$$

where $f_1(\omega)$ and $f_2(\omega)$ are the two polynomials defined as in Eqs. (7).

Definition 4: Let $S = \{\delta_i, i = 1, \ldots, \ell\}$. Also let S_1 and S_2 be two subsets of S, and $S_1 \cup S_2 = S$. If $S_1 \cap S_2 \neq \emptyset$ we define the restricted parameter vector d in $S_1 \cap S_2$, as follows:

$$d = (d_1, \ldots, d_m); \qquad d_i \in S_1 \cap S_2, \; i = 1, \ldots, m \tag{15}$$

The restricted parameter vectors associated with μ_1 and μ_2 are denoted by d_{s_1} and d_{s_2}, respectively.

The following lemma provides a sufficient condition for determining the real μ^* using $\mu_{12}(\omega)$, $\mu_1(\omega)$, and $\mu_2(\omega)$.

Lemma 3: Given $S = \{\delta_i, i = 1, \ldots, \ell\}$ and two subsets S_1 and S_2 of S with $S_1 \cup S_2 = S$, consider the following two cases: 1) $S_1 \cap S_2 = \emptyset$ and 2) $S_1 \cap S_2 \neq \emptyset$. By using the previous definitions of $\mu_1(\omega)$, $\mu_2(\omega)$, and $\mu_{12}(\omega)$, we have the following results:

Case 1: ($S_1 \cap S_2 = \emptyset$): The real μ^* can be found as

$$\mu^* = \sup_{\omega} \mu_{12}(\omega) = \max\left\{\sup_{\omega} \mu_1(\omega), \sup_{\omega} \mu_2(\omega)\right\} \tag{16}$$

Case 2: ($S_1 \cap S_2 \neq \emptyset$): If $\mu_1(\omega_c) = \mu_2(\omega_c)$ at some critical frequencies ω_c, and if the restricted parameter vectors in $S_1 \cap S_2$, associated with $\mu_1(\omega_c)$ and $\mu_2(\omega_c)$, become $d_{s_1} = d_{s_2}$, then the real μ^* is

$$\mu^* = \sup_{\omega} \mu_{12}(\omega) = \max_{\omega_c} \mu_1(\omega_c) = \max_{\omega_c} \mu_2(\omega_c) \tag{17}$$

Proof: The proof of this lemma is trivial and is therefore omitted here.

III. Sufficient Conditions for Corner Property

A characteristic polynomial, which has coefficients affine with respect to each uncertain parameter δ_i, is called a multilinearly uncertain polynomial. A dynamical system with such characteristic polynomial is called a multilinearly uncertain system or a system with multilinearly uncertain parameters.

If a system is described by the critical stability constraint of the form (5) with

$$\Delta = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_\ell)$$
$$m_i = 1 \quad \text{for} \quad i = 1, \ldots, \ell \tag{18}$$

then the system is a multilinearly uncertain system. However, not all multilinearly uncertain polynomials can be expressed in the form of (5) with (18); in many cases, we have Δ with repeated entries.

If the critical instability occurs at one of the corners of the parameter space hypercube, then the real parameter margin and the corresponding critical parameters can be easily determined using the following lemma.

Lemma 4: If the critical instability of the constraint (5) with possible repeated entries in Δ occurs at one of the corners of the parameter space hypercube, then

$$\kappa(\omega) = \left\{\max_{E \in \mathcal{E}} \rho[-EM(j\omega)]\right\}^{-1} \tag{19}$$

where $\rho(-EM)$ denotes the maximum real eigenvalue of $-EM$ and it is defined as zero if $-EM$ does not have real eigenvalues. Also, the corner matrix, denoted by E, is defined as

$$\mathcal{E} = \{E : E = \mathrm{diag}(e_i I_i), \; e_i = +1 \text{ or } -1, \; i = 1, \ldots, \ell\}$$

The real parameter margin κ^*, or real μ^*, is then determined as

$$\kappa^* \equiv 1/\mu^* := \inf_{\omega} \kappa(\omega)$$

The corresponding critical corner matrix E^* and critical corner vector e^* are, respectively, given by

$$E^* = \text{diag}(e_i^* I_i) \tag{20}$$

$$e^* = (e_1^*, e_2^*, \ldots, e_\ell^*) \tag{21}$$

Furthermore, the critical parameter vector δ^* can be determined as

$$\delta^* = \kappa^* e^*$$

Proof: See Ref. 13.

Remark: If $E \in \mathcal{E}$, then $-E \in \mathcal{E}$ and $\lambda(EM) = -\lambda(-EM)$ where $\lambda(EM)$ denotes the eigenvalues of EM. Thus, $\kappa(\omega)$ defined as (19) is always positive real.

We now give the following sufficient condition for the corner property of a multilinearly uncertain system.

Lemma 5: If a multilinearly uncertain polynomial $\phi(j\omega; \delta)$ is always real-valued at some frequency ω, then the critical instability at that frequency occurs at one of the corners of the parameter space hypercube $\kappa^*\mathfrak{D}$, where $\delta \in \kappa^*\mathfrak{D}$, or $-\kappa^* \le \delta_i \le \kappa^*$ $(i = 1, 2, \ldots, \ell)$.

Proof: See Refs. 13 and 14.

Corollary of Lemma 5: For a multilinearly uncertain system, the single-constraint real μs defined as in Definition 3 must attain their values at one of the corners of the parameter space hypercube $\kappa^*\mathfrak{D}$.

Theorem 1: At the degenerate frequencies defined as in Definition 2, the critical instability of a multilinearly uncertain system occurs at one of the corners of the parameter space hypercube.

Proof: At the degenerate frequencies, the two critical constraints (9) reduce to a single real-valued, linear or multilinear constraint, hence, we easily obtain this result from Lemma 5.

Corollary of Theorem 1: At $\omega = 0$, the critical instability occurs at one of the corners of the parameter space hypercube.

Proof: $M(j0)$ in the critical constraint (5) is a real-valued matrix. Hence, $\det[I + \Delta M(j0)]$ becomes real-valued and $\omega = 0$ is one of the degenerate frequencies. From Theorem 1, we obtain this corollary.

Theorem 2: Consider the two real critical constraints (9) with multilinearly uncertain parameters.

Case 1 of Lemma 3: The critical instability occurs at one of the corners of the parameter space hypercube.

Case 2 of Lemma 3: If $\mu_1(\omega)$ and $\mu_2(\omega)$ plots intersect at some frequencies ω_cs and if the restricted parameters vectors subject to $S_1 \cap S_2$, associated with $\mu_1(\omega_c)$ and $\mu_2(\omega_c)$, become $d_{s_1} = d_{s_2}$, then the critical instability occurs at one of the corners of the parameter space hypercube.

Proof: This theorem can be proved using Lemmas 3 and 5.

IV. Applications

A. Interval Polynomial

Consider a family of real polynomials

$$\phi(s, p) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \tag{22}$$

where each uncertain coefficient, $a_i \equiv p_i$, has a prescribed interval as

$$\underline{a}_i \le a_i \le \bar{a}_i, \qquad i = 1, 2, \ldots, n$$

The nominal values of a_i are

$$a_{0i} = (\bar{a}_i - \underline{a}_i)/2, \qquad i = 1, 2, \ldots, n$$

An interval polynomial with the normalized uncertain parameters δ_i will be used in the subsequent discussion.

A polynomial of the form (22) whose zeros lie on the open left-half s plane is called a Hurwitz polynomial. Kharitonov's theorem provides a simple way of checking whether a given polynomial whose coefficients have prescribed intervals is Hurwitz or not; however, it is not directly applicable to determining the size of the largest stable hypercube in the coefficient parameter space. In this section we employ $\mu_1(\omega)$ and $\mu_2(\omega)$ to determine the real parameter margin κ^* of an uncertain dynamical system described by an interval characteristic polynomial. An alternative proof of Kharitonov's theorem, which is based on the concept of $\mu_1(\omega)$ and $\mu_2(\omega)$, will be presented in the next section.

According to Lemma 2, an interval polynomial can be transformed into the two real constraints with the rank-one matrices $M_i(\omega)$, $i = 1, 2$. If n is an even integer, we have

$$M_i(\omega) = \frac{\alpha_i \beta_i^T(\omega)}{g_i(\omega)}, \qquad i = 1, 2$$

$$\Delta_1 = \text{diag}(\delta_2, \delta_4, \ldots, \delta_n), \qquad \Delta_2 = \text{diag}(\delta_1, \delta_3, \ldots, \delta_{n-1})$$

$$\alpha_1 = \alpha_2 = [1, 1, \ldots, 1]^T$$

$$\beta_1 = \beta_2 = [-\omega^{n-2}, \omega^{n-4}, \ldots, -\omega^2, (-1)^{n/2}]^T$$

$$g_1(\omega) = \omega^n - a_{02}\omega^{n-2} + \cdots + (-1)^{n/2} a_{0n}$$

$$g_2(\omega) = -a_{01}\omega^{n-2} + a_{03}\omega^{n-4} - \cdots + (-1)^{n/2} a_{0(n-1)}$$

Similar results can be obtained for the case of odd n.

Since $M_1(\omega)$ and $M_2(\omega)$ are rank-one matrices, we obtain the following theorem.

Theorem 3: The $\mu_1(\omega)$ and $\mu_2(\omega)$ of an interval polynomial of the form (22) attain their values at one of the corners of the parameter space hypercube and then can be expressed as

$$\mu_i(\omega) = -\frac{\alpha_i^T E_i \beta_i(\omega)}{g_i(\omega)}, \qquad i = 1, 2 \tag{23}$$

where

$$E_i = -\text{sgn}(g_i)\ \text{diag}\{\text{sgn}(\beta_{i1}), \text{sgn}(\beta_{i2}), \ldots, 1\}, \quad i = 1, 2 \tag{24}$$

and β_{ij} is the jth element of the column vector β_i, and $\text{sgn}(\cdot)$ denotes the signum function. The real parameter margin is then obtained as

$$1/\kappa^* \equiv \mu^* \equiv \sup_\omega \mu_{12}(\omega)$$

$$= \sup_\omega \left\{ -\frac{\alpha_1^T E_1 \beta_1(\omega)}{g_1(\omega)},\ -\frac{\alpha_2^T E_2 \beta_2(\omega)}{g_2(\omega)} \right\} \tag{25}$$

Proof: See Appendix.

B. Kharitonov's Theorem

Kharitonov's theorem is proven here using Theorem 3. Without loss of generality, we consider the case of even n.

The uncertain parameter sets corresponding to μ_1 and μ_2 of an interval polynomial are disjoint (i.e., $S_1 \cap S_2 = \emptyset$), and these disjoint parameter sets should have the same bound κ^*. Consequently, from Theorem 3, we have either

$$\mu^* = -\frac{\alpha_1^T E_1 \beta_1(\omega_c)}{g_1(\omega_c)} = \left| -\frac{\alpha_2^T E_2 \beta_2(\omega_c)}{g_2(\omega_c)} \right| \tag{26}$$

or

$$\mu^* = -\frac{\alpha_2^T E_2 \beta_2(\omega_c)}{g_2(\omega_c)} = \left| -\frac{\alpha_1^T E_1 \beta_1(\omega_c)}{g_1(\omega_c)} \right| \tag{27}$$

where E_i are defined as Eq. (24).

For the case of Eq. (26), the possible critical parameters are

$$(\delta_2, \delta_4, \ldots, \delta_n) = -\kappa^*[\text{sgn}(\beta_{11}), \text{sgn}(\beta_{12}), \ldots, 1]^T \text{sgn}(g_1)$$

$$(\delta_1, \delta_3, \ldots, \delta_{n-1}) = \pm\kappa^*[\text{sgn}(\beta_{21}), \text{sgn}(\beta_{22}), \ldots, 1]^T \text{sgn}(g_2)$$

which become

$$(\delta_2, \delta_4, \ldots, \delta_n) = \kappa^*[-1, 1, -1, \ldots, (-1)^{n/2+1}]^T \operatorname{sgn}(g_1)$$

$$(\delta_1, \delta_3, \ldots, \delta_{n-1}) = \pm\kappa^*[-1, 1, -1, \ldots, (-1)^{n/2+1}]^T \operatorname{sgn}(g_2)$$

For the case of Eq. (27), the possible critical parameters are

$$(\delta_1, \delta_3, \ldots, \delta_{n-1}) = -\kappa^*[\operatorname{sgn}(\beta_{21}), \operatorname{sgn}(\beta_{22}), \ldots, 1]^T \operatorname{sgn}(g_2)$$

$$(\delta_2, \delta_4, \ldots, \delta_n) = \pm\kappa^*[\operatorname{sgn}(\beta_{11}), \operatorname{sgn}(\beta_{12}), \ldots, 1]^T \operatorname{sgn}(g_1)$$

which become

$$(\delta_1, \delta_3, \ldots, \delta_{n-1}) = \kappa^*[-1, 1, -1, \ldots, (-1)^{n/2+1}]^T \operatorname{sgn}(g_2)$$

$$(\delta_2, \delta_4, \ldots, \delta_n) = \pm\kappa^*[-1, 1, -1, \ldots, (-1)^{n/2+1}]^T \operatorname{sgn}(g_1)$$

There are a total of 16 combinations of possible critical parameters, but only four of them are different from each other. The four corner vectors of the parameter space hypercube for possible critical instability at a corner are then obtained as

$$\delta^{(1)} = [-1, -1, 1, 1, \ldots, (-1)^{n/2+1}, (-1)^{n/2+1}]^T$$

$$\delta^{(2)} = [-1, 1, 1, -1, \ldots, (-1)^{n/2}, (-1)^{n/2+1}]^T$$

$$\delta^{(3)} = [1, -1, -1, 1, \ldots, (-1)^{n/2+1}, (-1)^{n/2}]^T$$

$$\delta^{(4)} = [1, 1, -1, -1, \ldots, (-1)^{n/2}, (-1)^{n/2}]^T$$

where $\delta = (\delta_1, \delta_2, \delta_3, \delta_4, \ldots, \delta_{n-1}, \delta_n)$. These are, in fact, Kharitonov's four corners for the case of even n.

C. Polytopic Polynomial

Consider a polynomial whose coefficients depend linearly on the normalized, perturbation parameter vector δ in $\kappa^*\mathfrak{D}$

$$\phi(s; \delta) = s^n + \sum_{i=1}^{n} a_i(\delta)s^{n-i} \tag{28}$$

where

$$a_i(\delta) = a_{0i} + \sum_{j=1}^{\ell} a_{ij}\delta_j, \qquad a_{ij} \text{ are constants}$$

According to Lemma 2, the critical constraints can be written as Eq. (9) with the rank-one matrices $M_1(\omega)$ and $M_2(\omega)$ such that

$$M_i(\omega) = \frac{\alpha_i(\omega)\beta_i^T(\omega)}{g_i(\omega)}, \qquad i = 1, 2$$

where $g_i(\omega)$ is a scalar function of ω, and α_i and β_i are column vectors that are functions of ω.

Because of the rank-one property of M_1 and M_2, we obtain the following result.

Theorem 4: The two single-constraint real μ measures of a polytopic polynomial of the form (28) will attain their values at one of the corners of the parameter space hypercube and can be expressed as

$$\mu_i(\omega) = -\frac{\alpha_i^T(\omega)E_i\beta_i(\omega)}{g_i(\omega)}, \qquad i = 1, 2 \tag{29}$$

where

$$E_i = -\operatorname{diag}\{\operatorname{sgn}(\alpha_{i1}\beta_{i1}), \ldots, \operatorname{sgn}(\alpha_{i\ell_i}\beta_{i\ell_i})\}\operatorname{sgn}(g_i)$$

for $i = 1, 2$, and α_{ij} and β_{ij} are, respectively, the jth elements of the column vectors α_i and β_i. If μ_1 and μ_2 intersect at some frequencies ω_cs and if $d_{s_1} = d_{s_2}$ at these frequencies, then the critical instability occurs at one of the corners of the parameter space hypercube.

Proof: The first part of this theorem is an extension of Theorem 3. The second part is an obvious application of Theorem 2.

D. Multilinearly Uncertain Polynomial

For a general case of a multilinearly uncertain polynomial, from the Corollary of Lemma 5, we know $\mu_1(\omega)$ and $\mu_2(\omega)$ will attain their values at one of the corners of the parameter space hypercube $\mathfrak{D}$ at any frequency ω. Hence we have the following theorem.

Theorem 5: The two single-constraint real μ measures will attain their values at one of the corners of the parameter space hypercube and can be expressed as

$$\mu_i(\omega) = \max_{E_i \in \mathcal{E}_i} \rho[-E_iM_i(\omega)], \qquad i = 1, 2$$

where

$$\mathcal{E}_i := \{E_i : E_i = \operatorname{diag}(e_jI_j),\ e_j = +1 \text{ or } -1\ \forall j\}$$

and $\rho(\cdot)$ denotes the maximum real eigenvalue of a matrix. If μ_1 and μ_2 intersect at some frequencies ω_cs and if $d_{s_1} = d_{s_2}$ at these frequencies, then the critical instability occurs at one of the corners of the parameter space hypercube.

Proof: The first part is an extension of Lemma 4. The second part is the direct application of Theorem 2.

V. Examples

A. Example 1: Vicino and Tesi[10]

Consider a polynomial whose coefficients are linearly dependent on uncertain parameters δ_1 and δ_2

$$\phi(s; \delta_1, \delta_2) = s^4 + (\delta_2 + 3)s^3 + (\delta_1 + 5.5)s^2 + (\delta_1 + \delta_2 + 4.5)s + 3\delta_1 - \delta_2 + 5.5$$

The two real critical constraints can be found as

$$\operatorname{Re}\{\phi(j\omega; \delta)\} = 0 \to f_1(\omega)\det[I_1 + \Delta_1 M_1(\omega)] = 0$$

$$\operatorname{Im}\{\phi(j\omega; \delta)\} = 0 \to f_2(\omega)\det[I_2 + \Delta_2 M_2(\omega)] = 0$$

where

$$f_1(\omega) = 1, f_2(\omega) = \omega, \qquad \Delta_1 = \Delta_2 = \operatorname{diag}(\delta_1, \delta_2)$$

and

$$M_i(\omega) = \frac{\alpha_i(\omega)\beta_i^T(\omega)}{g_i(\omega)}$$

with

$$\alpha_1 = \alpha_2 = [1, 1]^T$$

$$\beta_1 = [3 - \omega^2, -1]^T$$

$$\beta_2 = [1, 1 - \omega^2]^T$$

$$g_1(\omega) = \omega^4 - 5.5\omega^2 + 5.5$$

$$g_2(\omega) = -3\omega^2 + 4.5$$

And we obtain μ_1 and μ_2 as follows

$$\mu_1(\omega) = \frac{|3 - \omega^2| + 1}{|\omega^4 - 5.5\omega^2 + 5.5|}$$

$$\mu_2(\omega) = \frac{|1 - \omega^2| + 1}{|-3\omega^2 + 4.5|}$$

The critical corner matrices of $\mu_1(\omega)$ and $\mu_2(\omega)$ are

$$E_1 = -\operatorname{diag}\{\operatorname{sgn}(3 - \omega^2), -1\}\operatorname{sgn}(\omega^4 - 5.5\omega^2 + 5.5)$$

$$E_2 = -\mathrm{diag}\{1, \mathrm{sgn}(1-\omega^2)\}\mathrm{sgn}(-3\omega^2+4.5)$$

At the degenerate frequency of $\omega = 0$, we have

$$\mu(0) = \mu_1(0) = 1.375$$

By solving $\mu_1(\omega_c) = \mu_2(\omega_c)$ for ω_c, we find four critical frequencies. The corner matrices of $\mu_1(\omega)$ and $\mu_2(\omega)$ are the same only at $\omega_c = 1.4142$, and they are $E_1 = E_2 = \mathrm{diag}(1, -1)$. Since $\mu_1(1.4142) < \mu(0)$, the critical instability occurs at $\omega_c = 1.4142$. The critical corner matrix is $E^* = E_1 = E_2 = \mathrm{diag}(1, -1)$ and the critical corner vector is $e^* = (1, -1)$.

The real parameter margin κ^* becomes

$$\kappa^* = 1/\mu_{12} = 1/\mu_1(\omega_c) = 1/\mu_2(\omega_c) = 0.75$$

where $\omega_c = 1.4142$ and the critical parameter values are

$$(\delta_1^*, \delta_2^*) = \kappa^* e^* = \kappa^*(1, -1) = (0.75, -0.75)$$

B. Example 2: De Gaston and Safonov[3]

Consider a feedback control system consisting of a plant transfer function $G(s)$ and a compensator $K(s)$ given by

$$G(s) = \frac{p_1}{s(s+p_2)(s+p_3)}; \qquad K(s) = \frac{s+2}{s+10}$$

The uncertain parameters are described by

$$p_1 = 800(1+\epsilon_1), \qquad |\epsilon_1| \le 0.1$$
$$p_2 = 4+\epsilon_2, \qquad |\epsilon_2| \le 0.2$$
$$p_3 = 6+\epsilon_3, \qquad |\epsilon_3| \le 0.3$$

The closed-loop characteristic polynomial is

$$\phi(s;\epsilon_1,\epsilon_2,\epsilon_3) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

where

$$a_1 = 20 + \epsilon_2 + \epsilon_3$$

$$a_2 = 124 + 16\epsilon_2 + 14\epsilon_3 + \epsilon_2\epsilon_3$$

$$a_3 = 1040 + 800\epsilon_1 + 60\epsilon_2 + 40\epsilon_3 + 10\epsilon_2\epsilon_3$$

$$a_4 = 1600(1+\epsilon_1)$$

The uncertain parameters are normalized as

$$\delta_1 = \epsilon_1/0.1, \qquad \delta_2 = \epsilon_2/0.2, \qquad \delta_3 = \epsilon_3/0.3$$

The two real critical constraints can then be obtained as

$$\mathrm{Re}\{\phi(j\omega;\delta)\} = 0 \rightarrow f_1(\omega)\det[I_1 + \Delta_1 M_1(\omega)] = 0$$

$$\mathrm{Im}\{\phi(j\omega;\delta)\} = 0 \rightarrow f_2(\omega)\det[I_2 + \Delta_2 M_2(\omega)] = 0$$

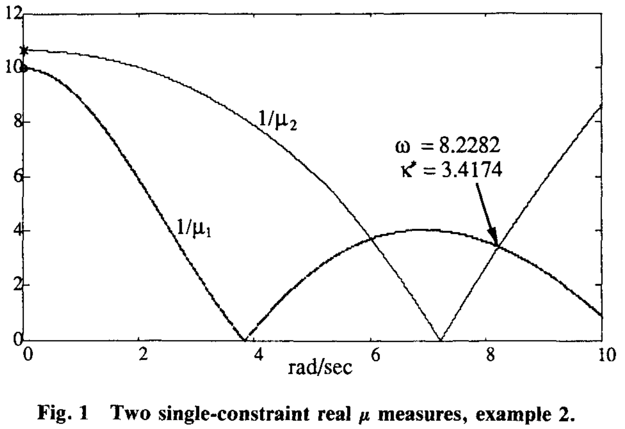

Fig. 1 Two single-constraint real μ measures, example 2.

where

$$f_1(\omega) = 1, \qquad f_2(\omega) = \omega$$

$$\Delta_1 = \Delta_2 = \Delta = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$$

$$M_i(\omega) = R_i(\omega) A_i^{-1}(\omega) L_i(\omega), \qquad i = 1, 2$$

and

$$R_1(\omega) = \begin{bmatrix} 1 & 0 \\ -3.2\omega^2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$R_2(\omega) = \begin{bmatrix} 1 & 0 \\ 12 - 0.2\omega^2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$L_1(\omega) = \begin{bmatrix} 160 & 1 & -4.2\omega^2 \\ 0 & 0 & 0.06\omega^2 \end{bmatrix}$$

$$L_2(\omega) = \begin{bmatrix} 80 & 1 & 12 - 0.3\omega^2 \\ 0 & 0 & 0.6 \end{bmatrix}$$

$$A_1^{-1}(\omega) = \begin{bmatrix} (1600 - 124\omega^2 + \omega^4)^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_2^{-1}(\omega) = \begin{bmatrix} (1040 - 20\omega^2)^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

As shown in Fig. 1, the plots of $1/\mu_1(\omega)$ and $1/\mu_2(\omega)$ vs ω intersect at two frequencies. We further find that only at $\omega_c = 8.2282$ are the corner directions for both $\mu_1(\omega)$ and $\mu_2(\omega)$ the same. Thus the critical corner matrices are

$$E_1^* = E_2^* = \mathrm{diag}(1, -1, -1)$$

and the real parameter margin is found as

$$\kappa^* = 1/\mu^* = 1/\mu_1(\omega_c) = 1/\mu_2(\omega_c) = 3.4174$$

where $\omega_c = 8.2282$. The corresponding critical corner vector is $e^* = (1, -1, -1)$ and the critical parameter values are

$$(p_1, p_2, p_3) = (1073.36, 3.3165, 4.9748)$$

C. Example 3: Ackermann's Multilinear Polynomial[8,15]

Consider a characteristic polynomial with ℓ uncertain parameters p_i given by

$$\phi(s;p) = \ell(\ell-1) + r^2 + 2(\ell+1)\sum_{i=1}^{\ell} p_i + 2\sum_{i<j}^{\ell} p_i p_j + \left(\ell + \sum_{i=1}^{\ell} p_i\right)s + \left(\ell + \sum_{i=1}^{\ell} p_i\right)s^2 + s^3$$

Here we consider a case with

$$\ell = 4, \qquad r = 0.5, \qquad p_{0i} = 4; \qquad (i = 1, \ldots, 4)$$

and let

$$p_i = p_{0i} + \delta_i; \qquad (i = 1, \ldots, 4)$$

The two real critical constraints can be found as follows:

$$\mathrm{Re}\{\phi(j\omega;\delta)\} = 0 \rightarrow f_1(\omega)\det[I_1 + \Delta_1 M_1(\omega)] = 0$$

$$\mathrm{Im}\{\phi(j\omega;\delta)\} = 0 \rightarrow f_2(\omega)\det[I_2 + \Delta_2 M_2(\omega)] = 0$$

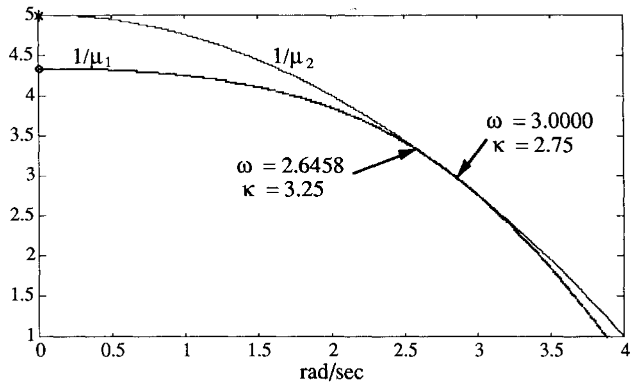

Fig. 2 Two single-constraint real μ measures, example 3.

where

$$f_1(\omega) = 1, \qquad f_2(\omega) = \omega$$

$$\Delta_1 = \mathrm{diag}(\delta_1, \delta_2, \delta_2, \delta_3, \delta_3, \delta_3, \delta_3, \delta_4)$$

$$\Delta_2 = \mathrm{diag}(\delta_1, \delta_2, \delta_3, \delta_4)$$

and

$$M_1(\omega) = R_1(\omega) A_1^{-1}(\omega) L_1(\omega)$$

$$M_2(\omega) = \frac{\alpha\alpha^T}{p_{01} + p_{02} + p_{03} + p_{04} + 4 - \omega^2}$$

$$R_1(\omega) = \begin{bmatrix} R_{11} & 0 \\ I_{4\times 4} & 0 \\ R_{31} & R_{32} \end{bmatrix}$$

where

$$R_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$R_{31} = [10 - \omega^2, 2, 2, 0]$$

$$R_{32} = [\ 2, 0, 0, 0]$$

$$A_1(\omega) = \begin{bmatrix} A_{11} & A_{12} \\ -p_{03} I_{4\times 4} & I_{4\times 4} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ -p_{01} & 1 & 0 & 0 \\ -p_{02} & 0 & 1 & 0 \\ 0 & -p_{02} & 0 & 1 \end{bmatrix}$$

$$a_{11} = 12 + r^2 - 4\omega^2 + (10 - \omega^2)(p_{01} + p_{02} + p_{03} + p_{04})$$

$$a_{12} = 2(p_{02} + p_{03} + p_{04})$$

$$a_{13} = 2(p_{03} + p_{04})$$

$$A_{12} = \begin{bmatrix} 2p_{04} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_1(\omega) = \begin{bmatrix} L_{11} & 1 \\ I_{7\times 7} & 0 \end{bmatrix}$$

$$L_{11} = [10 - \omega^2, 10 - \omega^2, 2, 10 - \omega^2, 2, 2, 0]$$

$$\alpha = [1, 1, 1, 1]^T$$

Since $M_2(\omega)$ is a rank-one matrix, we have

$$\mu_2(\omega) = \frac{\alpha^T\alpha}{|p_{01} + p_{02} + p_{03} + p_{04} + 4 - \omega^2|}$$

By Lemma 4, $\mu_1(\omega)$ can be found by checking corners, as follows

$$\mu_1(\omega) = \max_{E_1 \in \mathcal{E}_1} \rho[-E_1 M_1(\omega)]$$

As shown in Fig. 2, we find that the $1/\mu_1(\omega)$ and $1/\mu_2(\omega)$ plots intersect at two frequencies

$$\omega_{c1} = 2.6458 \text{ and } \omega_{c2} = 3.0000$$

At these frequencies, $\mu_1(\omega_c)$ and $\mu_2(\omega_c)$ have the following corner matrices:

$$E_1 = \mathrm{diag}(e_1, e_2, e_2, e_3, e_3, e_3, e_3, e_4)$$

$$E_2 = \mathrm{diag}(e_1, e_2, e_3, e_4)$$

$$e_1 = e_2 = e_3 = e_4 = 1$$

Then we have

$$\mu(\omega_{c1}) = \mu_{12}(\omega_{c1}) = \mu_2(\omega_{c1}) = 0.3636$$

$$\mu(\omega_{c2}) = \mu_{12}(\omega_{c2}) = \mu_2(\omega_{c2}) = 0.3077$$

The real parameter margin is found as

$$\kappa^* = 1/\mu^* = \min_{\omega_c} \{1/\mu_{12}(\omega_c)\} = 2.75$$

and the critical parameter values are

$$(p_1, p_2, p_3, p_4) = (1.25, 1.25, 1.25, 1.25)$$

D. Example 4: Chang et al.[6]

Consider the closed-loop characteristic polynomial of the second example of Ref. 6:

$$\phi(s; \delta_1, \delta_2, \delta_3) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 \tag{30}$$

where

$$a_1 = 10.4 - 0.3\delta_1 - 0.3\delta_2$$

$$a_2 = 38.14 - 2.31\delta_1 - 2.91\delta_2 + 0.45\delta_3 + 0.09\delta_1\delta_2$$

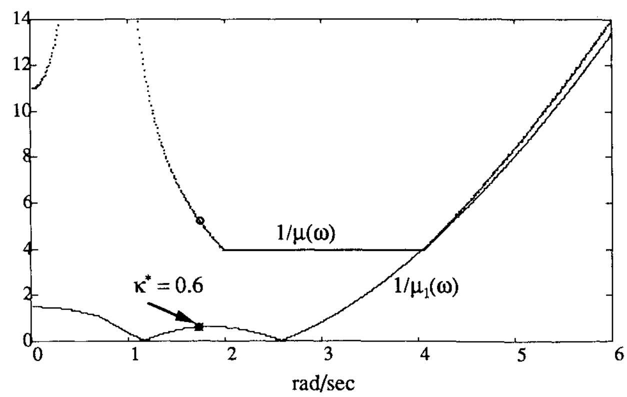

Fig. 3 Two single-constraint real μ measures, example 5.

$$a_3 = 58.12 - 5.97\delta_1 - 8.28\delta_2 + 1.74\delta_3 + 0.63\delta_1\delta_2 - 0.135\delta_2\delta_3$$

$$a_4 = 31.16 - 5.22\delta_1 - 6.84\delta_2 + 0.48\delta_3 + 1.08\delta_1\delta_2 - 0.27\delta_2\delta_3$$

One of the critical constraints has the form of

$$\Delta_1 = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$$

$$M_1(\omega) = R_1(\omega)A_1^{-1}(\omega)L_1(\omega)$$

and

$$R_1(\omega) = \begin{bmatrix} 2.31\omega^2 - 5.22 & 0.09\omega^2 - 1.08 \\ 1 & 0 \\ 0.48 - 0.45\omega^2 & 0.27 \end{bmatrix}$$

$$L_1(\omega) = \begin{bmatrix} 1 & 2.91\omega^2 - 6.84 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A_1^{-1}(\omega) = \begin{bmatrix} (\omega^4 - 38.14\omega^2 + 31.36)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

Hence the critical parameters will attain their values at the corner according to the Corollary of Theorem 1. The real μ^* can be found as

$$\mu^* = \mu_{12}(\omega_c) = \mu_1(\omega_c) = \max_{E \in \mathcal{E}} \rho[-EM_1(\omega_c)] = 0.2755$$

where $\omega_c = 0$. The real parameter margin is

$$\kappa^* = 1/\mu^* = 3.6297$$

and the corresponding critical corner matrix is

$$E^* = \mathrm{diag}(1, 1, 1)$$

For the nominal values of $\delta_i = 0$ $(i = 1, 2, 3)$, the corresponding critical parameter values are

$$(\delta_1^*, \delta_2^*, \delta_3^*) = (3.6297, 3.6297, 3.6297)$$

E. Example 5: Barmish et al.[12]

Consider a polynomial of the form

$$\phi(s; \delta_1, \delta_2) = s^4 + (4 - \delta_2)s^3 + (8 - 2\delta_1)s^2 + (12 - 3\delta_2)s + (9 - \delta_1 - 5\delta_2)$$

We use this example to show that the discontinuity in the real μ measure occurs at the degenerate frequencies and to further demonstrate that the real μ can be determined using one of the single-constraint real μ.

The two real critical constraints are

$$\mathrm{Re}\{\phi(j\omega; \delta)\} = 0 \rightarrow f_1(\omega) \det[I + \Delta_1 M_1(\omega)] = 0$$

$$\mathrm{Im}\{\phi(j\omega; \delta)\} = 0 \rightarrow f_2(\omega) \det[I + \Delta_2 M_2(\omega)] = 0$$

where

$$\Delta_1 = \mathrm{diag}(\delta_1, \delta_2), \qquad \Delta_2 = \delta_2$$

$$f_1(\omega) = 1, \qquad f_2(\omega) = 4(3 - \omega^2)\omega$$

and

$$M_1(\omega) = \frac{\alpha_1(\omega)\beta_1^T(\omega)}{g_1(\omega)}$$

with

$$g_1(\omega) = 9 - 8\omega^2 + \omega^4$$

$$\alpha_1(\omega) = [1, 1]^T$$

$$\beta_1(\omega) = [2\omega^2 - 1, -5]^T$$

and

$$M_2(\omega) = -1/4$$

At $\omega = \sqrt{3}$, the two real critical constraints reduce to one real critical constraint; i.e., $\omega = \sqrt{3}$ is the degenerate frequency. Since there are two constraints when $\omega \neq \sqrt{3}$, the two-constraint real μ measure, $\mu = \mu_{12}(\omega)$, becomes discontinuous at $\omega = \sqrt{3}$. The plots of $1/\mu$ and $1/\mu_1$ are shown in Fig. 3. As shown in Ref. 12, the critical instability occurs at $\omega = \sqrt{3}$ and it should occur at a corner. Thus the two single-constraint real μ measures are

$$\mu_1(\omega) = \frac{|2\omega^2 - 1| + 5}{|9 - 8\omega^2 + \omega^4|}, \qquad \mu_2(\omega) = 0.25$$

The real μ^* or the real parameter margin κ^* is then found as

$$\mu^* = 1/\kappa^* = \mu_1(\sqrt{3}) = 5/3$$

VI. Conclusions

The concept of two real critical constraints was introduced for determining the real parameter margins of dynamical systems with multilinearly uncertain parameters. Although the proposed approach is based on a sufficient condition for checking for critical instability only in the corner directions of the parameter space hypercube, the usefulness of the proposed approach has been demonstrated in this paper.

Appendix: Proofs of Lemmas and Theorems

Proof of Lemma 1: Let us first prove that a given polynomial $\phi(s; \delta)$ can be written as $\phi(s; \delta) = \phi(s; 0) \det[I + M(s)\Delta]$.

For a given rational square matrix $[I + M(s)\Delta(\delta)]$ with the elements of a parameter vector δ appearing in the numerators of each element of $[I + M(s)\Delta(\delta)]$, there exist polynomials $\phi(s; \delta)$, $\delta \in \mathfrak{D}$ and $\psi(s)$ such that

$$\det[I + M(s)\Delta(\delta)] = \frac{\phi(s; \delta)}{\psi(s)}$$

because of the property of the Smith-McMillan form of $[I + M(s)\Delta(\delta)]$ (e.g., see Theorem 2.3 in Ref. 16).

Since $\Delta(\delta) = \mathrm{diag}(\delta_i)$, $\Delta(0) = 0$. Consequently, we have

$$1 = \det[I] = \frac{\phi(s; 0)}{\psi(s)} \Rightarrow \psi(s) = \phi(s; 0)$$

and

$$\det[I + M(s)\Delta] = \frac{\phi(s; \delta)}{\phi(s; 0)} \tag{A1}$$

The existence of the matrices $M(j\omega)$ and Δ is obvious; however, the actual determination of $M(j\omega)$ and Δ is not straightforward. Without loss of generality, we use the following example to show how to form M and Δ for a given $\phi(s; \delta)$.

Consider a polynomial with coefficients that are multilinearly dependent on the four uncertain parameters δ_i

$$\phi(s; \delta) = a_0 + a_1\delta_1 + a_2\delta_2 + a_3\delta_3 + a_4\delta_4 + b_1\delta_1\delta_2 + b_2(\delta_1 + \delta_2)\delta_3 + b_3(\delta_1 + \delta_2)\delta_4$$

where a_i and b_i are functions of s.

We first transform this given polynomial into the determinant form of a 2×2 matrix, as follows:

$$\phi(s; \delta) = \det[G_0 + G_1\delta_1 + G_2\delta_2 + G_3\delta_3 + G_4\delta_4 + G_5\delta_1\delta_2]$$

where G_i are 2×2 matrices, defined as

$$G_0 = \begin{bmatrix} 0 & a_0 \\ -1 & 0 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} 0 & a_1 \\ 0 & 1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0 & a_2 \\ 0 & 1 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} b_2 & a_3 \\ 0 & 0 \end{bmatrix}, \quad G_4 = \begin{bmatrix} b_3 & a_4 \\ 0 & 0 \end{bmatrix}, \quad G_5 = \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix}$$

To change the term $G_5\delta_1\delta_2$ into a linear fractional form, we further transform the determinant into the determinant of a 4×4 matrix

$$\phi(j\omega;\, \delta) = \det[H_0 + H_1\delta_1 + H_2\delta_2 + H_3\delta_3 + H_4\delta_4] \tag{A2}$$

where H_i are 4×4 matrices defined as

$$H_0 = \begin{bmatrix} G_0 & 0 \\ 0 & I_{2\times 2} \end{bmatrix}, \quad H_1 = \begin{bmatrix} G_1 & G_5 \\ 0 & 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} G_2 & 0_{2\times 2} \\ -I_{2\times 2} & 0_{2\times 2} \end{bmatrix}, \quad H_3 = \begin{bmatrix} G_3 & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}$$

$$H_4 = \begin{bmatrix} G_4 & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}$$

Taking the singular value decomposition of H_i, we obtain

$$H_i = L_i R_i$$

with $\mathrm{rank}[L_i] = \mathrm{rank}[R_i] = r_i$.

Then Eq. (32) can be written as

$$\phi(s;\, \delta) = \det[H_0] \det[I + M(s)\Delta] \tag{A3}$$

where

$$M = RH_0^{-1}L$$

and

$$R = [R_1^T, R_2^T, \ldots, R_5^T]^T$$

$$L = [L_1, L_2, \ldots, L_5]$$

$$\Delta = \mathrm{diag}(\delta_i I_{r_i})$$

Finally we obtain

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_2 & a_3 & 0 & 0 \\ b_3 & a_4 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} a_1 & b_1 & 0 & a_2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

and

$$\Delta = \mathrm{diag}(\delta_1, \delta_1, \delta_2, \delta_2, \delta_3, \delta_4)$$

This form of M is not necessarily unique. Since we have

$$\det[H_0] = \det[G_0] = \det\begin{bmatrix} 0 & a_0 \\ -1 & 0 \end{bmatrix} = a_0 = \phi(s,\, 0)$$

Equation (A3) becomes Eq. (A1).

Proof of Lemma 2: Consider a constraint equation with the linearly dependent and/or linearly independent uncertain parameters of the form (4)

$$\phi(s;\, \delta) = \phi(s;\, 0) + \sum_{i=1}^{\ell} a_i(s)\delta_i = 0$$

where $\delta \in \mathfrak{D}$ denotes the uncertain parameter vector and $\mathfrak{D}$ denotes the parameter space hypercube. This polynomial equation can be separated as follows:

$$\mathrm{Re}[\phi(s;\, 0)] + \sum_{i=1}^{\ell} \mathrm{Re}[a_i]\delta_i = 0 \tag{A4}$$

$$\mathrm{Im}[\phi(s;\, 0)] + \sum_{i=1}^{\ell} \mathrm{Im}[a_i]\delta_i = 0 \tag{A5}$$

Equation (A4) can be rewritten as

$$0 = \det\begin{bmatrix} \mathrm{Re}[\phi(s;\,0)] + \Sigma_{i=1}^{\ell}\mathrm{Re}[a_i]\delta_i & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \det\left(\begin{bmatrix} \mathrm{Re}[\phi(s;\,0)] & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \mathrm{Re}[a_1] & 0 \\ 0 & 0 \end{bmatrix}\delta_1 + \cdots\right)$$

$$= \mathrm{Re}[\phi(s;\, 0)] \det[I + \tilde{M}\Delta]$$

where

$$\Delta = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_\ell)$$

$$\tilde{M} = \frac{1}{\mathrm{Re}[\phi(s;\,0)]} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} [\mathrm{Re}(a_1) \cdots \mathrm{Re}(a_\ell)]$$

Note that $\tilde{M}$ is a rank-one matrix. A similar result can be obtained for the other constraint equation (A5).

Proof of Theorem 3: From the Corollary of Theorem 2 we know that $\mu_1(\omega)$ and $\mu_2(\omega)$ will attain their values at one of the corners of the parameter space hypercube. Hence, according to Lemma 4, we have

$$\mu_i(\omega) = \max_{E_i \in \mathcal{E}_i} \rho[-E_i M_i(j\omega)], \qquad i = 1, 2$$

Since

$$\det\left[\lambda I + \frac{E_i\alpha_i(\omega)\beta_i^T(\omega)}{g_i(\omega)}\right] = \lambda + \frac{\beta_i^T(\omega)E_i\alpha_i(\omega)}{g_i(\omega)}$$

$$= \lambda + \frac{\alpha_i^T(\omega)E_i\beta_i(\omega)}{g_i(\omega)}$$

we have

$$\mu_i(\omega) = \max_{E_i \in \mathcal{E}_i} \left\{-\frac{\alpha_i^T(\omega)E_i\beta_i(\omega)}{g_i(\omega)}\right\}$$

Let

$$E_i = -\mathrm{sgn}(g_i)\, \mathrm{diag}\{\mathrm{sgn}(\beta_{i1}), \mathrm{sgn}(\beta_{i2}), \ldots, 1\}$$

then $\mu_i(\omega)$ will attain their maximum values.

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